On the Selberg integral of the $k-$divisor function and the $2k-$th moment of the Riemann zeta function

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1. Introduction.

We will link the two problems, estimating the $2k-$th moment of the Riemann $\zeta-$function (on the critical line $\sigma = \frac{1}{2}$)

$$I_k(T) \stackrel{def}{=} \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt$$

and finding non-trivial bounds for the, say, Selberg integral of the $k-$divisor function, $d_k(n)$ (that has Dirichlet series ζ^k)

$$J_k(x,h) \stackrel{def}{=} \int_{hx^{\varepsilon}}^{x} \Big| \sum_{t<n\le t+h} d_k(n) - M_k(t,h) \Big|^2 dt$$

where, say, $M_k(t,h)$ is the "expected value" of the (inner) sum. This gives d_k over the (say) "short interval" $[t, t+h]$ (as $h = o(t)$ $\forall t \in [hx^{\varepsilon}, x]$); here and in the sequel $\varepsilon > 0$ will be arbitrarily small, not the same at each occurrence.

Actually, Ivić already gave (in a paper to appear on JTNB for JAXXV Proc.) a non-trivial bound for $J_k(x,h)$ when the width of the s.i. (abbrev. short interval), namely $\theta := \frac{\log h}{\log x}$ is greater than $\theta_k \stackrel{def}{=} 2\sigma_k - 1$ (with σ_k Carlson's abscissa):

$$(\text{Ivić, JTNB}) \quad \theta > \theta_k \ \Rightarrow\ \exists \delta = \delta(k) > 0 \,:\, J_k(x,h) \ll \frac{xh^2}{x^{\delta}}$$

(see the trivial bound : $J_k(x,h) \ll xh^2(\log x)^c$, $c=c(k)$, here) For example, $\theta_3=\frac{1}{6}$, $\theta_4=\frac{1}{4}$, $\theta_5=\frac{11}{30}$, ... (from values of σ_k).

This result clearly gives non-trivial bounds for J_k, using $\zeta-$moments information (off the critical line, into the σ_k).

So, knowledge of the ζ implies knowledge of the d_k in A.A. (abbrev. almost all) the S.I. (short intervals).

$$\text{(See: } J_k \text{ non trivial} \Rightarrow \sum_{t<n\le t+h} d_k(n) \sim M_k(t,h), \text{ A.A.S.I.)}$$

However, we can also go in the opposite direction: if we have some kind of non-trivial information about the d_k, we can improve our knowledge (at least, on the $2k-$th moments) of the Riemann $\zeta-$function. Actually, this idea is due to Ivić, who gave a link between the "(AUTO-)CORRELATION" of d_k with "SHIFT-PARAMETER" a, namely

$$\mathcal{C}_k(a) \stackrel{def}{=} \sum_{n\le x} d_k(n)d_k(n+a) \qquad a\in\mathbb{N} \text{ (here } x\in\mathbb{N}, x\to\infty)$$

(the SHIFT is a positive integer, here; however, $\mathcal{C}_k(-a)$ is close enough to $\mathcal{C}_k(a)$; also, $\mathcal{C}_k(0)$ is relatively easy to compute). The problem of estimating d_k (AUTO-)CORRELATION, $\mathcal{C}_k(a)$, at least for a FIXED $a\neq 0$ ($a=0$ is "TRIVIAL") is called the $k-$ARY ADDITIVE DIVISOR PROBLEM ($k-$ARY ADD. DIV. PBM); the trivial case $k=1$ (with $\mathcal{C}_k(a)=x\ \forall a\in\mathbb{Z}$) and the "BINARY ADDITIVE DIVISOR PROBLEM", $k=2$, are the only two solved pbms, here.

Case $k=3$ is the TERNARY ADDITIVE DIVISOR PROBLEM (sometimes called "LINNIK PROBLEM", but Yu.V.Linnik has left many unsolved pbms !): some time ago, Vinogradov & Takhtadjan (right spelling ?) announced its solution but with, as yet, unfilled holes in their (extremely technical !) "proof".

This approach, actually, still suffers from our lack of information about $SL(\mathbb{Z},3)$; while our (enough good) state of the art about, instead, $SL(\mathbb{Z},2)$ allowed Ivić, Motohashi and Jutila to solve satisfactorily the binary additive divisor problem (but, different approaches work; however, with weaker remainders).

Thus, so far, no one has proved (for $k>2$), given $a\in\mathbb{N}$,

$$(*)_k \qquad \mathcal{C}_k(a) = xP_{2k-2}(\log x) + \Delta_k(x,a), \;\; \Delta_k(x,a) = o(x),$$

AS $x\to\infty$ (THE $k-$ARY ADDITIVE DIVISOR PROBLEM), not even for a single shift $a>0$ (already $k=2$ has delicate "$a-$UNIFORMITY" issues: following).
(MAIN TERM HAS $P_{2k-2}(\log x)$, A $2k-2$ DEG. POLY. IN $\log x$)

Here it comes into play the idea of Ivić (see his paper in Palanga 1996 Conference Proc.) of LINKING the estimate of the $2k-$TH MOMENT, $I_k(T)$, TO A SUM OF CORRELATIONS $\mathcal{C}_k(a)$ performed OVER a (THE SHIFT), up to (roughly, we avoid technicalities), say, $h := \frac{x}{T}$ (the S.I. comes in !)

In order to be more precise, we NEED also TO ABBREVIATE:

$$A \lll B \overset{def}{\Longleftrightarrow} \forall\varepsilon>0 \;\; A \ll_\varepsilon x^\varepsilon B$$

(with x, X or even T our "MAIN VARIABLES", indep.& $\to\infty$) I.E., the MODIFIED VINOGRADOV NOTATION $\lll$ allows us to IGNORE ALL THE ARBITRARILY SMALL POWERS; also, we'll say that the ARITHMETIC FUNCTION (A.F.) $f:\mathbb{N}\to\mathbb{R}$ is ESSENTIALLY BOUNDED, write $f\lll 1$, when $\forall\varepsilon>0$ $f(n)\ll_\varepsilon n^\varepsilon$ (as $n\to\infty$). For example, all the d_k ($\forall k\in\mathbb{N}$) are ess.bd :

$$\forall k\in\mathbb{N} \quad d_k \lll 1$$

whence they contribute individually small powers (ignored); Shiu estimates (see J_k triv.est.quoted above), a kind of Brun-Titchmarsh for (suitable multiplicative A.F., like) d_k, let these give, on average over (all) S.I., powers of log. By the way,

$$L := \log x \quad (\text{or } \ L := \log X, \ \text{ even } \ L := \log N)$$

is the abbreviation for the logarithm of our main variable.

For example, the (conjectured, $\forall k > 2$) M.T. of $(*)_k$ is

$$xP_{2k-2}(\log x) \ll_k xL^{2k-2} \lll x.$$

Here, it seems that the first to propose explicitly this form for $(*)_k$ is Ivić, who also gave explicitly the polynomial P_{2k-2}, that is essentially bounded (w.r.t. x). However, as we'll see in a moment, it depends, also, on the SHIFT $a > 0$.

We'll give now, avoiding technicalities, Ivić' s argument.

After some work (expand the square & mollify, take relevant ranges, ...) he gets that $I_k(T)$ is

$$I_k(T) = I_k''(T) + \mathcal{O}_\varepsilon(T^\varepsilon T)$$

with

$$I_k'' \stackrel{def}{=} \frac{1}{M} \sum_{a \le M^{1+\varepsilon}T^{-1}} \sum_{M < n \le M'} d_k(n) d_k(n+a) \int_{\frac{T}{2}}^{2T} \phi(t) e^{ita/n} dt,$$

where $M < M' \le 2M$, with $M \lll T^{k/2}$, say $h \lll M/T$, the smooth (i.e., C^∞) test-function ϕ has support into $]T/2, 2T[$, $\phi([3T/4, 4T/3]) \equiv 1$, and has good decay

$$\phi^{(R)}(t) \ll_R T^{-R}, \ \forall R \in \mathbb{N}.$$

We should keep in mind, here, that for fixed $k \in \mathbb{N}$ we seek

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt \lll T \quad (\text{“}2k\text{–th moment pbm”})$$

that (for $k > 2$) is our aim; in fact, English school gave first 2 cases: Ingham in 1926 asymptotics for $k = 1$ (not too difficult!) and $k = 2$ (for both, he gave only P_{2k-2} leading term, hence error $\log x$ better than main term); then, Heath-Brown in 1979 gave, for $k = 2$ (solved 2−add.div.pbm. using Weil's bound for Kloosterman sums), P_6 (not explicitly!) plus error $E_2 \lll T^{7/8}$.

Starting from '94 & '95, a series of Ivić & Motohashi papers (applying $SL(\mathbb{Z}, 2)$ considerations for the binary add.div.pbm.) gave $E_2 \lll T^{2/3}$ and, in mean-square, even $E_2 \lll \sqrt{T}$. (Here log-pows, not small pows!). Ivić explicited P_{2k-2} (when $k = 2$). Like the binary add.div.pbm., this is not the whole story !

The case $k = 3$, again (recall $\mathcal{C}_3(a)$ PBM), is unsolved. The bound

$$I_3(T) \lll T$$

is called THE “SIXTH MOMENT” PBM (actually, this is the weak version) & has a LINK (in Ivić, Proc. Cardiff 1996 Symposium) WITH the TERNARY ADDITIVE DIVISOR PBM.

Another interesting moment (Heath-Brown '79): $I_6 \ll T^2 L^c$.

One glimpse, $2k$−th pbm. ($k \le 2$, see Ivić & Motohashi, quoted): PREDICTED ASYMPTOTICS $I_k \sim C(k) T L^{k^2}$, $\forall k \ge 1$ (English school,again!) applying RANDOM MATRIX THEORY (Quantum Physics concepts inspiration!) in (2000s) seminal works of J. Keating & N. Snaith (at Bristol); many others (Conrey, Ghosh to name two). The RMT−ζ link originated (Dyson-Montgomery coffee-break) in 1972. (Sorry for brevity.)

Now on (that's why we mentioned weak $2k-$th pbm !) we can ignore (in bounds for I_k) all terms which are $\lll T$.
We give an idea of the polynomial, P_{2k-2}, given by Ivić, before to proceed. It's (see Palanga Conference Proc. for details)

$$P_{2k-2}(\log x) \stackrel{def}{=} \frac{1}{x}\int_0^x \sum_{q=1}^{\infty} \frac{c_q(a)}{q^2} R_k^2(\log t)dt,$$

with, say, $R_k(\log t) \stackrel{def}{=}$

$$\frac{C_{-k}(q)}{(k-1)!}\log^{k-1} t + \frac{C_{1-k}(q)}{(k-2)!}\log^{k-2} t + \ldots + \frac{C_{-2}(q)}{1!}\log t + C_{-1}(q)$$

depending on q, but not on a (this is vital); also, w.r.t. x, $R_k(\log t) \lll 1$ and this is very useful ! We'll see in a moment that the shape of these $C_j(q)$ is important only in case $q=1$. By the way, here $c_q(a)$ is the RAMANUJAN SUM, defined as ($\sum^*$ is over $q-$coprime js)

$$c_q(a) \stackrel{def}{=} \sum_{j(\bmod q)}^{*} e_q(ja) = \sum_{\substack{d|q \\ d|a}} d\mu\Big(\frac{q}{d}\Big)$$

Hence, say, $S(a) \stackrel{def}{=} \max(0, h-|a|)$ ($\hat{S}$ is Fejér's kernel) gives

$$\widehat{S}\Big(\frac{j}{q}\Big) \stackrel{def}{=} \sum_a S(a)e_q(ja) \ge 0 \Rightarrow \sum_a S(a)c_q(a) \ge 0$$

(see the link with J_k soon) and from the elementary, $\forall d \in \mathbb{N}$,

$$\sum_{\substack{a \\ a\equiv 0(d)}} S(a) = h + 2\sum_{b\le h/d}(h-db) = \frac{h^2}{d} + d\left\{\frac{h}{d}\right\}\left(1-\left\{\frac{h}{d}\right\}\right),$$

we get (apply $c_q(a)$, above), writing $\mathbf{1}_\wp = 1$ if $\wp$ holds, $= 0$ else:

$$\sum_a S(a)c_q(a) = \mathbf{1}_{q=1}h^2 + \sum_{d|q} d^2\mu\Big(\frac{q}{d}\Big)\left\{\frac{h}{d}\right\}\left(1-\left\{\frac{h}{d}\right\}\right).$$

(It is here evident $q=1$ greater importance.) Thus, (see Ivić):

$$(1)\qquad \sum_a S(a)xP_{2k-2}(\log x) = h^2\int_{hx^\varepsilon}^x R_k^2(1,\log t)dt + \text{TAILS},$$

where we mean, by "TAILS", remainders which are $\lll h^3$. Here, the part of $R_k(\log t)$ term with $q=1$ is, say, $R_k(1,\log t) \stackrel{def}{=}$

$$\frac{C_{-k}(1)}{(k-1)!}\log^{k-1}t + \frac{C_{1-k}(1)}{(k-2)!}\log^{k-2}t + \ldots + \frac{C_{-2}(1)}{1!}\log t + C_{-1}(1)$$

and equals the term $M_k(\log t)$ into the Selberg integral; as it should be, since (from an elementary version of the Linnik's Dispersion method), assuming $(*)_k$ with this P_{2k-2}, we get

$$(2)\qquad J_k(x,h) \sim \sum_a S(a)\mathcal{C}_k(a) - h^2\int_{hx^\varepsilon}^x M_k^2(\log t)\,dt$$

$$\sim \sum_a S(a)\Delta_k(x,a),$$

where $\sim$ means ignoring "TAILS" (see above) & "DIAGONALS", i.e. remainders $\lll xh$. We remark that BOTH THESE KIND OF ERRORS ARE NEGLIGIBLE, since they both contribute $\lll T$ to the final result for $I_k(T)$. (This can be easily seen.)

Then, due to I''_k expression, Ivić made a hypothesis about (avoiding technicalities) sums of $\Delta_k(x,a)$ (remainders into $(*)_k$ above), performed over the shift a, say G_k, which implies the bound $I_k(T) \lll T$ (for the same $k > 2$). NOW ON $k > 2$.

Of course, he doesn't need $(*)_k$ to hold INDIVIDUALLY $\forall a$ ($\le h$, here), but he observes that he's summing up, into G_k, WITHOUT the modulus over the remainder, $\Delta_k(x,a)$, so some $a-$cancellation can take place, here.

So far, he passes from an asymptotic formula $(*)_k$ to an $a-$averaged form of it, which is easier to prove (however, yet nobody has done it !).

Here, with applications in mind,

WE PASS FROM A SINGLE AVERAGE TO A DOUBLE AVERAGE

Building on his expression for I''_k, it's possible to make a less stringent hypothesis, to HAVE A MORE FLEXIBLE PROCEDURE for the remainders $\Delta_k(x,a)$.

We use, also, our previous works on the Selberg integral of the A.F. f (ESSENTIALLY BOUNDED & REAL) in order to let the Selberg integral of d_k, i.e. $J_k(x,h)$, come into play. (It is a kind of "DOUBLE AVERAGE" of $\Delta_k(x,a)$.)

Unfortunately, due to an exponential factor multiplying $d_k(n)d_k(n+a)$ into $\mathcal{C}_k(a)$ we can't get a link with $I_k(T)$ using only $J_k(x,h)$ (WITH $h \lll \frac{x}{T}$, $x \lll T^{k/2}$), but we need, also, to make an hypothesis on another double average of remainders $\Delta_k(x,a)$. We will give our Theorem and a sketch of the proof.

2. Result and Remarks.

THEOREM. *Let* $M < M' \le 2M$, $T^{1+\varepsilon} \le M \ll T^{k/2}$ *and* $H = M^{1+\varepsilon}/T$, *with double average* $\widetilde{G}_k = \widetilde{G}_k(M,T)$ *defined as*

$$\widetilde{G}_k \stackrel{def}{=} \sup_{M \le x \le M', t \le H} \left(\frac{1}{t} J_k(x,t) + \frac{1}{t} \left| \sum_{h \le t} \sum_{h < a \le t} \Delta_k(x,a) \right| \right).$$

Then for $k = 3, 4$ *and any fixed* $\varepsilon > 0$ *we have*

$$I_k(T) \ll_{k,\varepsilon} T^{1+\varepsilon} \left(1 + \sup_{T \lll M \ll T^{k/2}} \widetilde{G}_k(M,T)/M \right).$$

We remark that, in spite of the fact that Ivić 's Theorem holds $\forall k > 2$ (integer), we have some trouble in handling Selberg's integral TAILS, since they contribute to $\widetilde{G}_k$ as (in the sup above)

$$\lll \frac{1}{t} t^3 \lll H^2 \lll \frac{M^2}{T^2}$$

which gives to $I_k(T)$ a contribute (other sup above)

$$\lll T \left(\frac{M^2}{T^2} M^{-1} \right) \lll \frac{M}{T} \lll T^{k/2-1}$$

that is $\lll T$ ONLY WHEN $k/2 \le 2$, I.E. $k \le 4$ HERE.

We trust the possibility to have a link as above not only for the sixth & the eighth moment, but the TAILS arise naturally when applying the Linnik method and even a more careful analysis will almost surely not eliminate them ! While they're negligible for Selberg integral, not so for present application !

In order to give an idea of the proof, we'll avoid technics and highlight the ideas (few & elementary) involved.
First of all, since Ivić already has a sum of remainders, SAY

$$\sum_{a\le t}\Delta_k(x,a)$$

we simply let the ARITHMETIC MEAN appear (together with the other double sum, see $\widetilde{G}_k$ definition). Then, for this, see that

$$\frac{1}{t}\sum_{h\le t}\sum_{a\le h}\Delta_k(x,a)$$

(a kind of average of the above, something like C^1 process in Fourier series) can be expressed as (exchanging sums)

$$\frac{1}{t}\sum_{a\le t}(t-a+1)\Delta_k(x,a)=\frac{1}{t}\sum_{a\le t}(t-a)\Delta_k(x,a)+\frac{1}{t}\sum_{a\le t}\Delta_k(x,a)$$

which reduces to (using $\Delta_k(x,a)\lll x$)

$$\sim\frac{1}{t}\sum_{a\le t}(t-a)\Delta_k(x,a)\sim\frac{1}{2t}\sum_{0\le|a|\le t}(t-|a|)\Delta_k(x,a),$$

(+ diag.s & tails); and $S(a)=\max(t-|a|,0)$, $\forall 0\le|a|\le t\Rightarrow$

$$\sum_{0\le|a|\le t}(t-|a|)\Delta_k(x,a)\sim J_k(x,t)$$

we get the desired bound with the Selberg integral (& double average).

We remark, in passing, that "additional" double average can't be dispensed with.